УДК 811.111

FROM A BISECTRIX TO A HYPERBOLA

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Objectives:
1. to consider the properties of a bisectrix;
2. to find out new properties of a bisectrix;
3. relying on the properties of a bisectrix, to find a set of points possessing the same properties, but we take the following as the object:

• two points
• two straight lines
• a point and a straight line
• a point and a circle

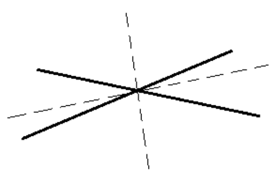
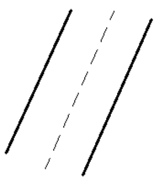
Goals:
1. to study the properties of bisectrix covered in the textbook and additional literature;
2. to study known sets of points in additional literature;
3. to study a set of points, equidistant to two objects (as the objects we take a point and a straight line; two, three straight lines; two, three points; a point and a circle)

Relevance:
1. Study of the bisectrix properties and discovery of new properties, will help to create the skill of research work:
   - Studying of additional literature;
   - Problem statement, hypothesizing and its proof;
   - Receiving of valid conclusions;
2. This research work has the practical significance for school-leavers concerning solution of tasks C of Unified State Exam.

The first stage of the research.

Let us find a set of points equidistant to two objects. As the objects we will take: straight lines; a point and a straight line; a point and a circle.

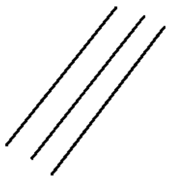
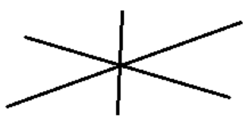
1. A set of points, equidistant to two straight lines.

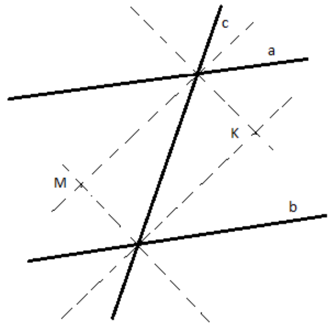
One straight line if these straight lines are parallel, or two mutually perpendicular straight lines if these straight lines meet.

2. Let us find a set of points, equidistant to three straight lines

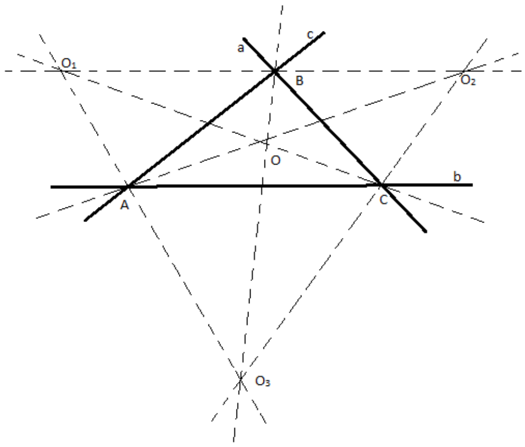
In the first instance the required set is empty.
In the second instance a point is the cross point of the straight lines.

In the third instance the required set represents two points: K and M.



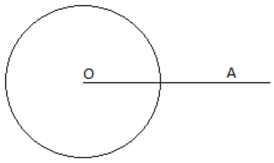
In the fourth instance the required set of points, in the case of this arrangement of straight lines, is four points: O – center of the inscribed circle; O_1, O_2, O_3 – centers of the excircles.



3. Let us take a point and a straight line as two objects.

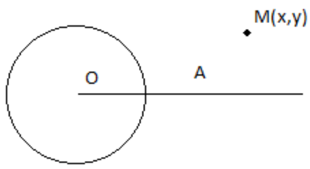
It is rather known fact – this is a parabola.

Let us take a point and a circle as two objects. Literature has no information on this set, therefore, the independent research was done, using the method of coordinates.



4. A set of points equidistant to a point and a circle

In the first instance when point A belonging to a circles with the center in point O and radius R. The required set is OA ray.



In the second instance point A lies outside a circle.

In the third instance point A lies within a circle.

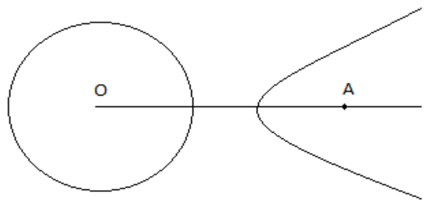
The distance from point M to a circle is OM – R.

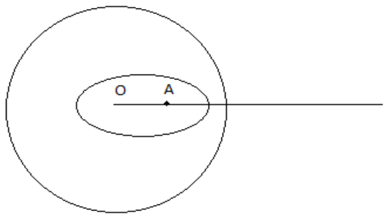
By condition OM – R = AM or R – OM = AM. Let us solve the problem analytically.

1. Let us write the equation. For this purpose we introduce a coordinate system: OX is precisely OA, point O – the coordinate origin. Then coordinates of point O (0,0), and coordinates of point A (a, 0).

2. $|OM - R| = AM$, turn to coordinates, $\left|\sqrt{x^2+y^2} - R\right| = \sqrt{(x-a)^2+y^2}$,therefore, $\sqrt{x^2+y^2} - R = \pm\sqrt{(x-a)^2+y^2}$. Let us transform this equation:
$x^2+y^2+R^2-2R\sqrt{x^2+y^2} = (x-a)^2+y^2$,$2ax+(R^2-a^2) = 2R\sqrt{x^2+y^2}$, square again, $4a^2x^2+4ax(R^2-a^2)+(R^2-a^2)^2 = 4R^2x^2+4R^2y^2$,
$4(a^2-R^2)x^2-4ax(a^2-R^2)-4R^2y^2+(a^2-R^2)=0$, $\quad where\ a^2-R^2 \neq 0, as\ a \neq R$

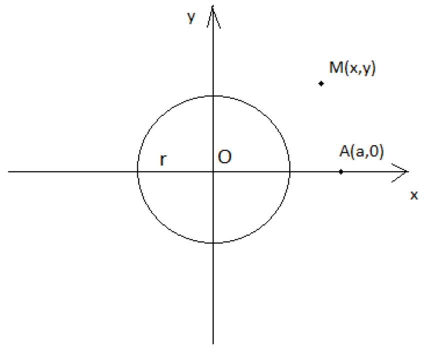
In case of a = R it is the equation of a straight line. Checking we verify that only the points of ray of OA fulfill condition (instance 1).
$x^2 - ax^2 - \frac{R^2}{a^2-R^2}y^2 + \frac{a^2-R^2}{4} = 0$, from here follows that $\left(x-\frac{a}{2}\right)^2 - \frac{R^2}{a^2-R^2}y^2 = \frac{R^2}{4}$
If a > R –we have the equation of a hyperbola.
If a < R – we have equation of an ellipse.





The second stage of the research.

Let us find a set of points, for each of which the relation of distances to two objects is a constant value (we take two points, a point and a straight line, a point and a circle as objects).
1. We take a point and a straight line as two objects.
In this case we have an ellipse and a parabola again.
Literature says that these point and straight line are called a focus and a directrix.
A parabola, ellipse and hyperbola are a set of points:
1) Equidistant to two objects: parabola (a point and a straight line), hyperbola and ellipse (a point and a circle)
2) On the other hand, the set of points, the relation of distances from each of which to two straight lines is a constant value, if these objects are a point and a straight line,
- if t=1 – a parabola,
- if t<1 – an ellipse,
- if t>1 – a hyperbola



3) Let us look at the second instance if we take a point and a circle. Let us introduce coordinate system, as in the first part of the work.
Let $\frac{MA}{\rho(M,окр)} = t$.

We have $\frac{\sqrt{(x-a)^2+y^2}}{\sqrt{x^2+y^2}-r}=t$. After squaring, we have the equation (after the first squaring) $2rt^2\sqrt{x^2+y^2}=(t^2-1)x^2-2ax+(t^2-1)x^2+(r^2t^2-a^2)$.

1) t=1 (this case is considered above). Depending on the position of point A we have OA ray, an ellipse, a hyperbola.
2) t≠1

After the next squaring, we have the equation of the fourth order. To see the line given by this equation, we take the specific numbers r=1, t=2, a=3. If t>1, a>r, we have the following equation: $4\sqrt{x^2+y^2}=3x^2-6x+3y^2-5$

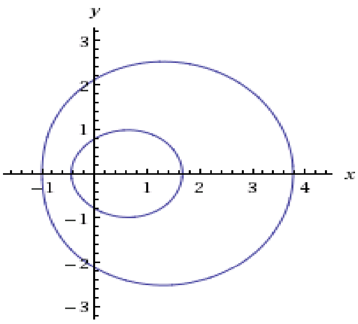
$$16(x^2+y^2)=9x^4+36x^2+9y^4+25-36x^3+18x^2y^2-30x^2-36xy^2+60x-30y^2$$

$$9x^4+18x^2y^2+9y^4-36x^3-36xy^2+6x^2-30y^2+60x+25=16x^2+16y^2$$

$$9x^4+18x^2y^2+9y^4-36x^3-36xy^2+10x^2-46y^2+60x+25=0$$

$$9(x^2+y^2)^2-36x(x^2+y^2)-10x^2-46y^2+60x+25=0$$

If a=r, t≠2. We cannot set up the received equation independently. Therefore, we use the means of computer algebra for this purpose.



Conclusion

The properties of a bisectrix of angle are considered. From all the properties the following ones are emphasized:
- a sets of points equidistant to the sides of an angle,
- it divides the opposite side of an angle into parts proportional to adjacent sides of the angle.

The task was to find a set of a plane points equidistant to two objects. The objects were: 2 straight lines, 3 straight lines; a point and a straight line; a point and a circle. The following was found out: a set of points equidistant to 2 and 3 straight lines is straight lines and points, depending on the arrangement of straight lines; a set of points equidistant to straight line and a point is a parabola. The information was found in the literature, therefore, only the conclusion was made.

Otherwise, no information about a set of points equidistant to a point and a circle, was found, therefore, the research was done using the method of coordinates. For this purpose the theory of lines of the second order was studied. The set can be presented as a ray, hyperbola or ellipse, depending on the position of a point relative to a circle.

The second stage of the research was to find of a set of points for each of which the relation of distances to two objects is a constant value (we take two points, a point and a straight line, a point and a circle as objects). When a point and a straight line were objects an ellipse and a hyperbola were got. The most difficult was to find this set when a point and a circle were taken as objects. The equation was determined, and the means of computer algebra were used for this purpose.